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ON THE NONLINEAR INTEGRO-DIFFERENTIAL EQUATION OF THE THEORY OF UNSTEADY FILTRATION WITH UNKNOWN BOUNDARY

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The problem of contraction of the boundary oil content in an oil-bearing stratum and of the motion of a heavy incompressible fluid with nonlinear condition at the free surface can be reduced to the Cauchy problem for a nonlinear integro-differential equation [1, 2]. Solvability of the Cauchy problem for equations of that type with smooth or discontinuous initial conditions is considered. A linearization method is developed on the example of problems of dispersion of ground water mound and of lowering the level of ground water by drainage. The result of linearization is either a Fredholm equation of the second kind or an equation that can be reduced to it. Solvability of that equation is proved and the linearization error estimated.

1. Let us consider the problem of determination in region $\Omega = \{(x, t): -\infty < x < +\infty,\ 0 \leqslant t \leqslant T\}$ the solution $h(x, t)$ of the nonlinear integro-differential equation

$$h_t - \frac{1}{\pi} \int_{-\infty}^{\infty} [k_0 + h_t(\xi, t)]\, K_0^-(x - \xi; h, \eta)\, d\xi - B_0 h = 0 \qquad (1.1)$$

$$K^-_0(x; h, \eta) = [x h_x(x, t) - (h - \eta)]\,[x^2 + (h - \eta)^2]^{-1}, \quad \eta = h(\xi, t)$$

which satisfies the condition

$$h(x, t)\,|_{t=+0} = h_e(x) \qquad (1.2)$$

where $B_0 h$ is some generally nonlinear operation (depending on the form of B_0, (1.1) and (1.2) correspond to different problems of unsteady filtration with nonlinear conditions at the free surface; examples of such problems appear in [1, 2] and below) and k_0 is a constant parameter.

Let $C_L^{1+\nu, 1+\nu}(\Omega)$ be a set of functions absolutely integrable together with their derivatives which are continuous with respect to t ($t \in [0, T]$) and satisfy Hölder's condition with exponent ν, $0 < \nu_0 \leqslant \nu \leqslant 1$ with respect to x; $C_L^{\nu, 0}(\Omega)$ is a set of functions that have the same properties as the derivatives of functions from the set $C_L^{1+\nu, 1+\nu}$.

Lemma. Let $h \in C_L^{1+\nu, 1+\nu}(\Omega)$ and $B_0 h \in C_L^{\nu, 0}(\Omega)$, then the operator of the left-hand side of (1) maps the set of functions belonging to $C_L^{1+\nu, 1+\nu}(\Omega)$ into the set $C_L^{\nu, 0}(\Omega)$.

To prove this we represent the integral with kernel K_0^- in the form

$$\int_{-\infty}^{\infty} K_0^-(x-\xi;h,\eta)\,d\xi = -\int_{-\infty}^{\infty} D_0(h)\left[1-\frac{D_1^2(h)}{1+D_1^2(h)}\right]d\xi \tag{1.3}$$

$$D_0(h) = \xi^{-2}[h(x,t)+\xi h_x(x,t)-h(x+\xi,t)],$$
$$D_1 = \xi^{-1}[h(x,t)-h(x+\xi,t)]$$

It is sufficient to show that the integral of D_0 belongs to the set $C_L^{\nu,0}$. This is achieved by splitting the integral into two parts which correspond to segments $|\xi| < 1$ and $|\xi| > 1$.

Using formula (1.3) we transform Eq. (1.2) to

$$h_t(x,t)+\frac{k_0}{\pi}\int_{-\infty}^{\infty} D_0(h)\,d\xi = B_1 h \tag{1.4}$$

$$B_1 h = B_0 h - \frac{1}{\pi}\int_{-\infty}^{\infty} h_t(\xi,t)K_0^-\,d\xi + \frac{k_0}{\pi}\int_{-\infty}^{\infty}\frac{D_0(h)D_1^2(h)}{1+D_1^2(h)}\,d\xi$$

where in conformity with the lemma $B_1 h \in C_L^{\nu,0}$.

To analyze problem (1.2), (1.4) we consider the subsidiary problem of finding function $h(x,t)$ that satisfies the equation

$$h_t(x,t)+\frac{k_0}{\pi}\int_{-\infty}^{\infty} D_0(h)\,d\xi = f(x,t) \tag{1.5}$$

and condition (1.2) if $h_e \in C_L^{1+\nu}$, $h_e^\circ \in C_L^{\nu}$, $f(x,t) \in C_L^{\nu,0}$, where

$$\Phi h_e^\circ(x) = |\alpha|\,\Phi h_e, \quad \Phi h \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h(x,t)\,e^{-i\alpha x}\,dx$$

T h e o r e m. When $h_e \in C_L^{1+\nu}$, $f(x,t) \in C_L^{\nu,0}$ and $h_e^\circ \in C_L^{\nu}$ then the solution of problem (1.2), (1.5) belongs to set $C_L^{1+\nu,1+\nu}$ and is determined by formula

$$h(x,t) = h_e(x) * \psi(x,t) + Bf \tag{1.6}$$

$$Bf \equiv \Phi^{-1}\int_0^t \exp[-k_0|\alpha|(t-\tau)]\,\Phi f(x,\tau)\,d\tau$$

$$h_e * \psi = \int\limits_{-\infty}^{\infty} h_e(x-\xi)\,\psi(\xi, t)\,d\xi, \quad \psi(x, t) = \frac{1}{\pi}\frac{k_0 t}{x^2 + k_0^2 t^2}$$

Solution (1.6) is determined using the Fourier transformation. From the equality

$$Bf = tf(x, \ t^\circ) * \psi(x, \ t_0), \quad t^\circ = t(1-\theta), \quad t_0 = \theta t$$
$$0 < \theta < 1$$

we obtain the relationships

$$h_x = h_e'(x) * \psi(x, t) + \frac{2}{k_0 \pi \theta}\int\limits_0^{\infty} [f(x + \lambda k_0 t_0, t^\circ) - \tag{1.7}$$
$$f(x - \lambda k_0 t_0, t^\circ)]\frac{\lambda d\lambda}{(\lambda^2+1)^2}$$
$$h_t = -k_0 h_e^\circ * \psi(x, t) + f(x, t) - k_0 t f(x, t^\circ) * \frac{\partial \psi}{\partial t}(x, t_0)$$

From (1.6) and (1.7) we have the following estimates:

$$|h| \leqslant \sup_x |h_e| + t \sup_x |f| \tag{1.8}$$
$$\int\limits_{-\infty}^{\infty} |h|\,dx \leqslant \int\limits_{-\infty}^{\infty} |h_e|\,dx + t\int\limits_{-\infty}^{\infty} |f|\,dx$$
$$|h_x| \leqslant \sup_x |h_e'| + \frac{2}{k_0 \pi \theta}\sup_\lambda |f(x + \lambda k_0 t_0, t^\circ) - f(x - \lambda k_0 t_0, t^\circ)|$$
$$\int\limits_{-\infty}^{\infty} |h_t|\,dx \leqslant k_0 \int\limits_{-\infty}^{\infty} |h_e^\circ|\,dx + 2\int\limits_{-\infty}^{\infty} |f(x, t^\circ)|\,dx$$
$$|h_x(x, \ t) - h_x(y, \ t)| \leqslant [c_1 + c_2(1 + 1/(\pi k_0)]\,|x-y|^\nu$$
$$|h_t(x, \ t) - h_t(y, \ t)| \leqslant [k_0 c_3 + c_2(1/\pi + 3/2)]\,|x-y|^\nu$$

where c_1, c_2, and c_3 are constants in Hölder's conditions for functions $h_e f$ and h_e°, respectively. This proves the theorem.

C o r o l l a r y. The elements of set BB_2h, where B_2 is any operator that transforms set $h \in C_L^{1+\nu, 1+\nu}$ into $C_L^{\nu, 0}$, belong to set $C_L^{1+\nu, 1+\nu}$ (B is determined by formula (1.6)).

According to Theorem 1 and its corollary problem (1.2), (1.4) (or (1.1)) is equivalent to the equation

$$h - h_0 - BB_1 h = 0, \qquad h_0 = h_e * \psi(x, \ t) \tag{1.9}$$

whose any solution $h \in C_L^{1+\nu, 1+\nu}$.

R e m a r k. In the subsequent determination of sets $C_L^{\nu, 0}$ and $C_L^{1+\nu, 1+\nu}$ we substitute the condition

$$|h|, \quad |h_x(x, t)|, \quad |h_t| \leqslant \text{const } (1 + |x|^{1+\beta})^{-1}, \ \beta > 0$$

for the condition of absolute integrability.

We introduce the norm

$$\|h\|_{C_*} = \max_t \sup_x e^{-Mt} [(1 + |x|^{1+\beta}) (|h| + \lambda_1 |h_x| + \lambda_2 |h_t|) + \lambda_3 \|h\|_\nu] \tag{1.10}$$

$$\|h\|_\nu = \sup_{x,y} \frac{|h_x(x, t) - h_x(y, t)| + |h_t(x, t) - h_t(y, t)|}{|x - y|^\nu}$$

where M and λ_j $(j = 1, 2, 3)$ are positive parameters which will be determined below.

Let us assume the existence in some sphere $\|h - h_0\| \leqslant R$ of derivatives B_0' and B_0'' of the nonlinear operator B_0 which map set $C_L^{1+\nu, 1+\nu}$ into $C_L^{\nu, 0}$, and that

$$|B_0'(h)\Delta h| \leqslant c_4 \|\Delta h\|, \quad |B_0''\Delta h \Delta' h| \leqslant c_5 \|\Delta h\| \|\Delta' h\| \tag{1.11}$$

It is now possible to consider the nonlinear operation $P(h)$ which corresponds to the left-hand side of (1.9) and maps sphere $\|h - h_0\| \leqslant R$ into space C_*. In that sphere operation $P(h)$ has the first and second order derivatives

$$P'(h)\Delta h = \Delta h - BB_1'(h)\Delta h, \quad P''\Delta h \Delta' h = -BB_1''(h)\Delta h \Delta' h$$

It can be readily shown that, owing to the linearity of operations D_0 and D_1, $B_1'(h)\Delta h$ and $B_1''(h)\Delta h \Delta' h$ belong to set $C_L^{\nu, 0}$.

The nonlinear equation $P(h) = 0$ is analyzed using the Newton—Kantorovich method [3,4]. Function h_0 is taken as the initial approximation of Newton's process and an estimate is obtained of the norm of the inverse operator $[P'(h_0)]^{-1} = [E - A]^{-1}$ which yields the solution of the integro-differential equation

$$u - Au = -P(h_0), \quad P(h_0) = -BB_1(h_0) \tag{1.12}$$

$$Au \equiv BB_1'(h_0)u, \quad u = \Delta h = h_1 - h_0, \quad h_0 = h_e * \psi$$

where $h_1(x, t)$ is the sought first approximation and E is a unit vector. Thus the solution $u(x, t)$ of Eq. (1.12) determines the first approximation of Newton's process for Eq. (1.9) (or problem (1.1), (1.2)).

Since $B_1'(h)\Delta h \in C_L^{\nu, 0}$, hence in conformity with the corollary of Theorem 1 it is possible to use estimates (1.8). Then, taking into account the uniform boundedness of functions u, u_x, and u_t, and of equalities

$$\partial / \partial x BB_1'u = tBB_1'u\,(x,\ t_0) * \psi_x'(x,\ t^\circ)$$
$$\partial / \partial t BB_1'u = BB_1'u - k_0 tBB_1'u\,(x,\ t_0) * \psi_t'\,(x,\ t^\circ_r)$$

we obtain

$$e^{-Mt}\left(|Au| + \lambda_1\left|\frac{\partial}{\partial x}Au\right| + \lambda_2\left|\frac{\partial}{\partial t}Au\right| + \lambda_3\|Au\|_\nu\right) \leqslant$$
$$c_6\left[\frac{1-e^{-Mt}}{M} + \frac{4\lambda_1}{\pi k_0} + \lambda_2\left(\frac{3}{2}+\frac{1}{\pi}\right) + \frac{\lambda_3 c_\nu}{R}\right]\max_t \sup_x |ue^{-Mt}|$$

where c_ν is the sum of constants in Hölder's conditions.

The influence of $1 + |x|^{1+\beta}$ leads to the substitution of another constant for c_6. We have

$$\|A\| \leqslant c_7\left[\frac{1-e^{-Mt}}{M} + \frac{4\lambda_1}{\pi k_0} + \left(\frac{3}{2}+\frac{1}{\pi}\right)\lambda_2 + \lambda_3 c_8\right] \equiv \varkappa \qquad (1.13)$$

The estimate

$$\|BB_1''\| \leqslant c_7(1-e^{-Mt}) / M + c_9(\lambda_1+\lambda_2+\lambda_3) \equiv \varkappa_1$$

can be similarly obtained.

We assume M to be fairly large and λ_j fairly small so that $\varkappa < 1$. Let the input function $h_0(x)$ be such as to ensure the inequality $\|P(h_0)\| \leqslant (1-\varkappa)R$, then according to the principle of contractive mapping there exists in the sphere $\|u\| \leqslant R$ a unique solution of Eq. (1.12), and

$$\|[E-A]^{-1}\| \leqslant (1-x)^{-1} \qquad (1.14)$$

T h e o r e m 2. Let the following conditions:

1) the nonlinear operation B_0 and its derivatives B_0' and B_0'' map the set of functions $h \in C_L^{1+\nu,\,1+\nu}$ into $C_L^{\nu,\,0}$ and the estimate (1.11) hold,

2) function $h_0(x) \in C_L^{1+\nu}$ is such that

$$\|P(h_0)\| \leqslant {}^1/_2(1-\varkappa)R \qquad (1.15)$$

3) parameter M ensures the validity of inequalities

$$\varkappa_0 R\,(1-\varkappa_0)^{-1} < 1, \quad \varkappa\,|_{\lambda_j=0} = \varkappa_0 < \varkappa < 1 \qquad (1.16)$$

are satisfied, then the unique solution of the Cauchy problem can be obtained by using the Newton's process.

Parameters λ_j are selected to as to transform (1.16) into nonrigorous inequalities. There exists then a unique solution of Eq. (1.12) and all conditions of Kantorovich's theorem [3,4], from which follows Theorem 2, are satisfied. Note that condition (1.15) and estimate (1.14) ensure the validity of inequalities

$$(1-\sqrt{1-2\eta_0})\,\eta_0^{-1} \leqslant R\,(\|[P'(h_0)]^{-1}\|\,\|P(h_0)\|)^{-1} \leqslant$$
$$(1+\sqrt{1-2\eta_0})\,\eta_0^{-1}$$
$$\eta_0 = {}^1/_2\varkappa_1 R\,(1-\varkappa)^{-1}$$

in which the equality sign can only apply when $\eta_0 = {}^1/_2$.

Let us consider the case of discontinuous initial conditions. Let $h_s(x) \in L$ be a step function. It is then possible to consider, without loss of generality, the single-step function

$$h_s(x) = d = \text{const}, \ |x| < 1, \quad h_s(x) = 0, \quad |x| > 1$$

The limit value of the derivative of $h_t(x, +0)$ can be obtained by assuming that Eq. (1.1) is valid up to $t = +0$, except at points $|x| = 1$, i.e.

$$u - \frac{1}{\pi}\int_{-\infty}^{\infty}[k_0 + u(\xi)]\,K(x,\xi)\,d\xi = f(x), \quad |x| \neq 1$$

$$u = h_t(x, +0), \quad f(x) = Bh\,|_{t=+0}$$

$$K(x,\xi) = \begin{cases} 0, \ |\xi| < 1, \quad |x| < 1; \quad -d\,[1+(x-\xi)^2]^{-1}, \quad |\xi| > 1, \quad |x| < 1 \\ d\,[1+(x-\xi)^2]^{-1}, \quad |\xi| < 1, \quad |x| > 1; \quad 0, \quad |\xi| > 1, \quad |x| > 1 \end{cases}$$

If the Fourier formula holds for function $f(x)$, the application of the Fourier transformation the solution of the latter equation is obtained in the explicit form

$$u = f_2 + f^- + u_1 * \psi(x, 1), \quad |x| > 1 \tag{1.17}$$

$$u = f_1 + f^+ + (f_1 + f^+) * \sum_{k=1}^{\infty}(-1)^k\psi(x, 2k) -$$
$$f_2 * \sum_{k=0}^{\infty}(-1)^k\psi(x, 2k+1), \quad |x| < 1$$
$$f^+ = f(x)\,\chi(1-|x|), \quad f^- = f(x)\chi(|x|-1)$$

$$\chi(x) = 1, \ x > 0; \quad \chi(x) = 0, \ x < 0; \quad \psi(x, a) = a/[\pi(x^2+a^2)]$$
$$f_1(x) = -dk_0/\pi^{-1}[\pi - \operatorname{arctg}(x+1) + \operatorname{arctg}(x-1)]\chi(1-|x|)$$
$$f_2(x) = dk_0/\pi^{-1}[\operatorname{arctg}(x+1) - \operatorname{arctg}(x-1)]\chi(|x|-1)$$

Obviously

$$\lim_{t\to+0} h_x(x,\ t) = 0, \quad |x| \neq 1 \tag{1.18}$$

It is then possible to represent Eq. (1.1) in the form (1.4) only when $t > 0$ and, consequently, if $f(x,\ t) \in C_L^{\gamma,0}$ in the region with deleted points $(x, t) = (\pm 1, 0)$, then unlike in (1.17) and (1.18), the behavior of derivatives of h_x and h_t of the solution of problem (1.2), (1.4) near the discontinuity point $|x| = 1$ is of order $O(t^{-1})$ when $t \to +0$. The left-hand side of (1.9) has finite limits when $t \to +0$ ($|x| \neq 1$), but its individual terms have singularities of the indicated type. Owing to this in the estimates for h_x and h_t the constants are replaced by functions of t which tend to infinity when $t \to +0$. Omitting the proof, we note that solvability of this problem is proved by the norm (1.10) where $t \in [T_0,\ T]$, $T_0 > 0$ only for $t \in [T_0,\ T]$.

As an example of B_0 we consider the operator

$$B_0 h = -\frac{1}{\pi}\int_{-\infty}^{\infty} [k_0 + h_t(\xi,\ t)]\, K_0^+(x-\xi;\ h,\ \eta)\, d\xi - \tag{1.19}$$
$$q(t)\, K_0(x;\ h,\ c_0 + h_\infty) - k_0$$

$$K_0^+(x;\ h,\ \eta) = [xh_x - (h + \eta - 2h_\infty)][x^2 + (h + \eta - 2h_\infty)^2]^{-1}$$

$$K_0(x,\ h,\ \eta) = K_0^-(x;\ h,\ \eta) + K_0^+(x;\ h,\ \eta)$$

$$h_\infty = \lim_{|x|\to\infty} h_u, \quad h_u = h + h_\infty, \quad 0 < c_0 < 1 - \varepsilon_0, \quad \varepsilon_0 > 0$$

It is assumed that when $q \not\equiv 0$, $h_e(x) \equiv 0$ and $h_u(x,\ +0) = h_\infty = 1$, where h_u is the ground water level measured from the waterproof stratum.

Here $q \equiv 0$ relates to the problem of dispersion of a ground water mound in a stratum of finite thickness ($B_0 h \equiv 0$ corresponds to its dispersion in an infinitely thick stratum), while $q \not\equiv 0$ corresponds to the problem of the fall of ground water level produced by a sink of intensity $q(t)$ located at point $(x,\ z) = (0,\ c_0)$ of the motion region $\{(x,\ z){:}\ -\infty < x < +\infty,\ 0 < z < h(x,\ t)\}$ [2].

In the first case ($q \equiv 0$) conditions 1) of Theorem 2 is satisfied for any finite radius R, $R > \exp(-MT) \sup |h_e|$, and in the second for $R \leqslant (1 - c_0 - \varepsilon_0) \exp(-MT)$. The necessity of the latter constraint is due to the form of $K_0(x,\ h;\ c_0 + h_\infty)$ and is related to the loss of solution uniqueness for the ground water level in proximity of the tubular drain contour which is replaced by a line sink [5]. In that case with $h_u(0,\ t) \to c_0$ the assumption that $z = h_u(x,\ t)$ belongs to Liapunov curves [2] is no longer valid.

Condition (1.15) can be satisfied by a suitable selection of functions $h_e(x)$ (q

$\equiv 0)$ or $q(t)$ $(q \not\equiv 0)$.

2. Linearization of the nonlinear integro-differential equation. Equation (1.1) with allowance for (1.19) is of the form

$$h_t - \frac{1}{\pi}\int_{-\infty}^{\infty}[k_0 + h_t(\xi, t)]\,K_1(x-\xi; h, \eta)\,d\xi + \tag{2.1}$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{h_t(\xi, t)\,d\xi}{(x-\xi)^2+4} - q(t)\,K_0(x; h, c_0 + h_\infty) = 0$$

$$K_1(x-\xi; h, \eta) = K_0(x-\xi; h, \eta) + 2/[(x-\xi)^2+4]$$

An approximate solution of problem (1.2), (2.1) can be obtained by the linearization method. For this the time interval $[0, T]$ is subdivided by points $0 = t_0 < t_1 < \ldots < t_N = T$ in N parts, and K_0 and K_1 at the small intervals $(t_m, t_{m+1}]$ are replaced by their values for $t = t_m - 0$ $(m = 1, 2, \ldots, N-1)$

$$K_1(x-\xi; h, \eta) \approx K_1(x-\xi; h, \eta)\,|_{t=t_m-0} \equiv K_1(x, \xi; t_m)$$

$$K_0(x; h, c_0 + h_\infty) \approx K_0(x; h(x, t_m - 0), \ c_0 + h_\infty) \equiv K_0(x, t_m)$$

For $m = 0$ functions h and η in K_0 and K_1 are replaced by their initial values. Then, instead (2.1) we have the linear equation

$$u_{m+1}(x, t) - \frac{1}{\pi}\int_{-\infty}^{\infty}u(\xi, t)\,K_1(x, \xi; t_m)\,d\xi + \tag{2.2}$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{u_{m+1}(\xi, t)\,d\xi}{(x-\xi)^2+4} = f_0(x, t; t_m), \quad t \in (t_m, t_{m+1}]$$

$$f_0(x, t; t_m) = \frac{k_0}{\pi}\int_{-\infty}^{\infty}K_1(x, \xi; t_m) + q(t)\,K_0(x, t_m)$$

$$u_{m+1}(x, t) = h_t(x, t), \quad t \in (t_m, t_{m+1}]$$

If we assume that the Hölder exponent $\nu > 1/2$ and that the Fourier formula holds for f_0, the Fourier transformation makes it possible to reduce (2.2) to the Fredholm equation of the second kind

$$u_{m+1}(x, t) = \frac{1}{\pi}\int_{-\infty}^{\infty}u_{m+1}(\xi, t)\,K(x, \xi; t_m)\,d\xi = f(x, t_m) \tag{2.3}$$

$$K = K_1 + K_2, \quad K_2 = K_1(x, \xi; t_m) * g(x)$$

$$g(x) = \sum_{k=1}^{\infty}(-1)^k\,\psi(x, 2k), \quad f = f_0 + f_0 * g$$

The homogeneous equation that corresponds to (2.3) has only a trivial solution, since owing to the uniqueness of the Fourier transformation it is equivalent to Eq. (2.2). The homogeneous equation which corresponds to the latter is at the limit the same as Neumann's homogeneous external equation for expanding regions tending to the band

$$\{(x, z): -\infty < x < +\infty, \ -h_u(x, t) < z < h_u(x, t)\}, \ t \in (t_m, t_{m+1}]$$

This implies the triviality of solution of the homogeneous equation and the unique solvability of (2.3).

Note that for a fairly smooth input function $h_e(x)$ the solution of (2.3) is $m = N-1$ and $u_N \in C_L^{1+\nu, 0}$ and $u_N' \in C_L^{\nu, 0}$, hence the Fourier formula is valid for function $f_0(x; t; t_m)$, $m = 1, 2, \ldots, N-1$.

The linearization error

$$\varepsilon_{m+1} = u(x, t) - u_{m+1}(x, t), \quad t \in (t_m, t_{m+1}], \ u = h_t$$

where $h(x, t)$ is the exact solution of Eq. (2.1) and satisfies the equation of the form ((2.3) with kernel $K = K_1(x - \xi; h_{m,m}; \eta_{m,m})$ and the right-hand side of $f = f_\varepsilon \Delta h_m$, where f_ε denotes some linear operation on Δh_m. That equation is similar to (2.2) and reduces to a Fredholm equation of the second kind; moreover we have the equality

$$\Delta h_m = (t - t_m) u(x, t_m^*) + \sum_{k=1}^{m} (t_k - t_{k-1}) \varepsilon_k(x, t_{k-1}^\circ), \quad m \geqslant 1$$

$$\Delta h_0 = tu(x, t_0^\circ), \quad t_{k-1} < t_{k-1}^\circ < t_k, \quad k = 1, 2, \ldots, m$$

$$h_{m,m} = \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} u_{k+1}(x, \tau)\, d\tau$$

from which, on the assumption that $h \in C_L^{1+\nu, 0}$ we find that the linearization error ε_m $(m = 1, 2, \ldots, N)$ is of order $O(\Delta t)$ where $\Delta t = \max\limits_{1 \leqslant k \leqslant m} (t_k - t_{k-1})$. The difference between the proposed linearization method and the earlier modification of Euler's method [1] applies to integro-differential equations of this type is similar to the difference between the results obtained with implicit and explicit difference schemes applied to the solution of equations of the parabolic type.

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